Electromagnetic Theory

1.1 Electromagnetic Wave

Here we investigate the dynamical properties of the electromagnetic field by deriving a set of equations which are alternatives to the Maxwell equations. It turns out that these alternative equations are wave equations, indicating that electromagnetic waves are natural and common manifestations of electrodynamics.

Maxwell's microscopic equations are:

$\nabla . E = \frac{\rho}{\varepsilon_0}$ (Gauss's law in electrostatics) (1)

$\nabla \times E = -\frac{\partial B}{\partial t}$ (Faraday's law) (2)

$\nabla . B = 0$ (Gauss's law in Magnetostatics) (3)

$\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} + \mu_0 j(t, x)$ (Ampère's law) (4)

and can be viewed as an axiomatic basis for classical electrodynamics. In particular, these equations are well suited for calculating the electric and magnetic fields E and B from given, prescribed charge distributions ρ(t,x) and current distributions j(t, x) of arbitrary time- and space-dependent form. However, as is well known from the theory of differential equations, these four first order, coupled partial differential vector equations can be rewritten as two uncoupled, second order partial equations, one for E and one for B.

1.2 Maxwell's Equations

We are now able to collect the results from the above considerations and formulate the equations of classical electrodynamics valid for arbitrary variations in time and space of the coupled electric and magnetic fields E(t, x) and B(t, x). The equations are

$\nabla . E = \frac{\rho}{\varepsilon_0}$ (1)

$\nabla \times E = -\frac{\partial B}{\partial t}$ (2)

$\nabla . B = 0$ (3)

$$\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} + \mu_0 j(t, x) \qquad (4)$$

In these equations ρ(t, x) represents the total, possibly both time and space dependent, electric charge, i.e., free as well as induced (polarization) charges, and j(t, x) represents the total, possibly both time and space dependent, electric current, i.e., conduction currents (motion of free charges) as well as all atomistic (polarization, magnetization) currents. As they stand, the equations therefore incorporate the classical interaction between all electric charges and currents in the system and are called Maxwell's microscopic equations. Another name often used for them is the Maxwell-Lorentz equations. Together with the appropriate constitutive relations, which relate ρ and j to the fields, and the initial and boundary conditions pertinent to the physical situation at hand, they form a system of well-posed partial differential equations which completely determine E and B.

1.3. Maxwell Stress Tensor

The derivation starts with a calculation of the total force due to electromagnetic fields on the charges and currents within some volume $\mathcal{V}$. From the Lorentz force law, we have

$$\mathbf{F} = \int_{\mathcal{V}} \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B})\, d^3\mathbf{r} \qquad (1)$$

$$= \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d^3\mathbf{r} \qquad (2)$$

We can think of the integrand as a force density, or force per unit volume $\mathbf{f}$:

$$\mathbf{f} \equiv \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \qquad (3)$$

We can express this entirely in terms of fields by using Maxwell's equations:

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} \qquad (4)$$

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad (5)$$

So we get

$$\mathbf{f} = (\epsilon_0 \nabla \cdot \mathbf{E})\,\mathbf{E} + \left[\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right] \times \mathbf{B} \qquad (6)$$

We now need to do a bit of vector calculus gymnastics. From the product rule

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \qquad (7)$$

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$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{8}$$

Combining these two we get

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \tag{9}$$

$$= \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{10}$$

We can insert this into 6 and while we're at it, we can add on a term $\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\,\mathbf{B}$. This is always zero because $\nabla \cdot \mathbf{B} = 0$, but it gives the equation a symmetry that will be useful in a minute. We get for the force density:

$$\mathbf{f} = \epsilon_0 (\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \tag{11}$$

$$= \epsilon_0 (\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \tag{12}$$

$$\epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{13}$$

Now another identity from vector calculus says

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{14}$$

If $\mathbf{A} = \mathbf{B} = \mathbf{E}$, we get

$$\nabla \left(E^2\right) = 2\mathbf{E} \times (\nabla \times \mathbf{E}) + 2(\mathbf{E} \cdot \nabla)\mathbf{E} \tag{15}$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla \left(E^2\right) - (\mathbf{E} \cdot \nabla)\mathbf{E} \tag{16}$$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2}\nabla \left(B^2\right) - (\mathbf{B} \cdot \nabla)\mathbf{B} \tag{17}$$

Putting this into 12 we get

$$\mathbf{f} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \tag{18}$$

$$\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E})$$

$$= \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \tag{19}$$

$$\frac{1}{2} \nabla \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) + \epsilon_0 (\mathbf{E} \cdot \nabla) \mathbf{E} + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

$$= \epsilon_0 [(\nabla \cdot \mathbf{E}) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{E}] + \frac{1}{\mu_0} [(\nabla \cdot \mathbf{B}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{B}] - \tag{20}$$

$$\frac{1}{2} \nabla \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

It might not seem that we're making any progress, since the equations just get longer with each alteration. However, we can now introduce the Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$ which is a 3×3 matrix with components defined by

$$\boxed{T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)} \tag{21}$$

Note that the tensor is symmetric: $T_{ij} = T_{ji}$. If we define the scalar product of the tensor with an ordinary vector to be another vector:

$$\left[\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}} \right]_j = \sum_i a_i T_{ij} \tag{22}$$

where the subscript j indicates the jth component of the resulting vector, then the divergence is

$$\left[\nabla\cdot\overleftrightarrow{\mathbf{T}}\right]_j = \sum_i \partial_i T_{ij} \tag{23}$$

$$= \epsilon_0 \sum_i \left((\partial_i E_i) E_j + E_i (\partial_i E_j) - \frac{1}{2}\delta_{ij}\partial_i E^2 \right) + \frac{1}{\mu_0}\sum_i \left((\partial_i B_i) B_j + B_i (\partial_i B_j) - \frac{1}{2}\delta_{ij}\partial_i B^2 \right) \tag{24}$$

$$= \epsilon_0 \left((\nabla\cdot\mathbf{E}) E_j + (\mathbf{E}\cdot\nabla) E_j - \frac{1}{2}\partial_j E^2 \right) + \frac{1}{\mu_0}\left((\nabla\cdot\mathbf{B}) B_j + (\mathbf{B}\cdot\nabla) B_j - \frac{1}{2}\partial_j B^2 \right) \tag{25}$$

Comparing this with 20, we see that we can write $\mathbf{f}$ in terms of $\overleftrightarrow{\mathbf{T}}$ and the Poynting vector as

$$\boxed{\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0 \frac{\partial \mathbf{S}}{\partial t}} \tag{26}$$

The total force on the volume is then

$$\mathbf{F} = \int_{\mathcal{V}} \mathbf{f} d^3\mathbf{r} \tag{27}$$

$$= \int_{\mathcal{V}} \left(\nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0 \frac{\partial \mathbf{S}}{\partial t} \right) d^3\mathbf{r} \tag{28}$$

From the formula 23 for the divergence, we can see that the vector resulting from the divergence has as its components the divergences of each column of $\overleftrightarrow{\mathbf{T}}$. Therefore we can apply the divergence theorem to the first term in the integrand to get

$$\boxed{\mathbf{F} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{S} d^3\mathbf{r}} \tag{29}$$

where $\mathcal{S}$ is any surface that encloses only the charges and currents within $\mathcal{V}$.